

# ATTRACTING SEQUENCES OF HOLOMORPHIC AUTOMORPHISMS THAT AGREE TO A CERTAIN ORDER

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**ABSTRACT.** The basin of attraction of a uniformly attracting sequence of holomorphic automorphisms that agree to a certain order in the common fixed point, is biholomorphic to  $\mathbb{C}^n$ . We also give sufficient estimates how large this order has to be.

## 1. INTRODUCTION

The systematic study of basins of attraction of holomorphic automorphisms goes back to works of Sternberg [Ste57] and Dixon and Esterle [DE86]. The first complete result<sup>1</sup> was obtained by Rosay and Rudin [RR88] who showed that the basin of attraction of a holomorphic automorphism with an attracting fixed point is always biholomorphic to  $\mathbb{C}^n$ . The question can be generalized to sequences of holomorphic automorphisms  $f_j: \mathbb{C}^n \rightarrow \mathbb{C}^n$  with a common attracting fixed point  $z_0 \in \mathbb{C}^n$ .

**Definition 1.1.** Let  $f_j: \mathbb{C}^n \rightarrow \mathbb{C}^n$ ,  $j \in \mathbb{N}$ , be a sequence of holomorphic self-maps. Their *basin of attraction in*  $z_0 \in \mathbb{C}^n$  is defined to be

$$\Omega_{f_j}^{z_0} := \left\{ z \in \mathbb{C}^n : \lim_{j \rightarrow \infty} f_j \circ f_{j-1} \circ \cdots \circ f_1(z) = z_0 \right\}$$

A counterexample by Fornæss [For04] shows that this basin of attraction of holomorphic automorphisms with a common fixed point  $z_0$  does in general not need to be biholomorphic to  $\mathbb{C}^n$ .

Therefore the question whether this basin of attraction is biholomorphic to  $\mathbb{C}^n$  is usually considered for automorphisms that satisfy the following uniform attraction property:

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<sup>1</sup>Rosay and Rudin [RR88] remarked that the result of Dixon and Esterle [DE86] relies on a statement of Reich [Rei69] which in turn had a gap in its proof. The earlier result of Sternberg [Ste57] deals only with an automorphism whose differential in the attracting fixed point is diagonal and has no special elements – for the definition of special elements, see Section 3.

**Definition 1.2.** We call a sequence of holomorphic automorphisms  $f_j: \mathbb{C}^n \rightarrow \mathbb{C}^n$ ,  $j \in \mathbb{N}$ , *uniformly attracting* in a point  $z_0 \in \mathbb{C}^n$ , if there exist real numbers  $0 < r, s < 1$  and  $\delta > 0$  such that

$$\forall j \in \mathbb{N} \forall z \in B_\delta \quad s\|z - z_0\| < \|f_j(z) - z_0\| < r\|z - z_0\|$$

where  $B_\delta := \{z \in \mathbb{C}^n : \|z\| < \delta\}$ .

It was shown by Fornæss and Stensønes [FS04] that if  $\Omega_{f_j}^{z_0}$  is biholomorphic to  $\mathbb{C}^n$  for any sequence of uniformly attracting holomorphic automorphisms, then this would give a positive answer to the stable manifold conjecture of Bedford. Their result has drawn a lot of interest and several positive partial results have been obtained so far. In particular we want to mention a result of Wold [Wol05, Theorem 4] that has been generalized by Sabiini and then further improved by Peters and Smit [PS15].

**Theorem 1.3.** [Sab10, Sab16] *Let  $0 < r, s < 1$ ,  $0 < \delta$ , and let  $p \in \mathbb{N}$  such that  $r^p < s$ . Then for any uniformly attracting sequence of holomorphic automorphisms  $f_j: \mathbb{C}^n \rightarrow \mathbb{C}^n$  with*

$$\forall j \in \mathbb{N} \forall z \in B_\delta \quad s\|z - z_0\| < \|f_j(z) - z_0\| < r\|z - z_0\|$$

*such that*

$$(1) \quad \frac{\partial^{|\alpha|} f_j}{\partial z^\alpha}(z_0) = 0 \text{ for all multi-indices } \alpha \in \mathbb{N}_0^n \text{ with } 2 \leq |\alpha| \leq p-1$$

*the basin of attraction is biholomorphic to  $\mathbb{C}^n$ .*

This contains the result of Wold [Wol05, Theorem 4] for  $p = 2$  where the condition (1) is empty. Recently, this condition was further improved in dimension  $n = 2$  by Peters and Smit [PS15] using the method of so-called adaptive trains.

Another positive result was obtained by Peters [Pet07] when all the uniformly attracting automorphisms  $f_j$  are uniformly close to a given automorphism:

**Theorem 1.4** ([Pet07]). *Given a holomorphic automorphism  $f_1: \mathbb{C}^n \rightarrow \mathbb{C}^n$  with 0 as attractive fixed point there exists  $\varepsilon > 0$  such that for any sequence of holomorphic automorphisms  $f_j: \mathbb{C}^n \rightarrow \mathbb{C}^n$  fixing 0 and satisfying  $\|f_1 - f_j\|_{B_1} < \varepsilon$ , the basin of attraction is biholomorphic to  $\mathbb{C}^n$ .*

In our paper we want to consider the situation when the higher partial derivatives in the common fixed point do not necessarily vanish, but instead agree up to a certain order  $q$ .

**Theorem 1.5.** *Given  $0 < r, s < 1$ ,  $0 < \delta$ , and a holomorphic automorphism  $f_1: \mathbb{C}^n \rightarrow \mathbb{C}^n$  there exists a number  $q \in \mathbb{N}$  such that for any uniformly attracting sequence of holomorphic automorphisms  $f_j: \mathbb{C}^n \rightarrow \mathbb{C}^n$  with*

$$\forall j \in \mathbb{N} \forall z \in B_\delta \quad s\|z - z_0\| < \|f_j(z) - z_0\| < r\|z - z_0\|$$

*that agree in the fixed point  $z_0$  modulo terms of order  $q$ , i.e.*

$$(2) \quad \frac{\partial^{|\alpha|} f_1}{\partial z^\alpha}(z_0) = \frac{\partial^{|\alpha|} f_j}{\partial z^\alpha}(z_0)$$

*for all multi-indices  $\alpha \in \mathbb{N}_0^n$  with  $1 \leq |\alpha| \leq q-1$ , the basin of attraction is biholomorphic to  $\mathbb{C}^n$ .*

*Let  $p \in \mathbb{N}$  be such that  $r^p < s$ . Then we have the following estimates for  $q$ :*

(1) *If each of the derivatives  $d_{z_0} f_j$ ,  $j \in \mathbb{N}$ , is normal, then*

$$q \leq \frac{\ln \left( \sum_{i=1}^{(p-1)(n-1)^2} Q(i) \left( \frac{n!}{s^n} \left( 1 + r \sum_{m=2}^{p-1} n^2 Q^2(m) \left( \frac{\sqrt{n}}{\min\{1, \delta\}} \right)^m \right)^{n-1} \right)^i \right)}{\ln \frac{1}{r}} + 1$$

*independent of  $f_1$ ,*

(2) *In case of dimension  $n = 2$  and if each of the derivatives  $d_{z_0} f_j$ ,  $j \in \mathbb{N}$ , is normal, then*

$$q \leq \frac{\ln \left( \sum_{i=1}^{p-1} (i+1) \left( \frac{2}{s^2} \left( 1 + 2rp \left( \frac{\sqrt{2}}{\min\{1, \delta\}} \right)^{p-1} \right) \right)^i \right)}{\ln \frac{1}{r}} + 1 \in O(p^2)$$

*independent of  $f_1$ ,*

*where  $Q(m)$  denotes the number of multi-indices in  $n$  variables of order  $m$ .*

## 2. THE ROSAY–RUDIN FRAMEWORK

In this section we state and prove the key proposition which goes back to Rosay and Rudin [RR88, Appendix] for the basin of attraction of a single automorphism. Several special cases of this proposition have been used in the literature, but to the authors' knowledge, it has never been stated as a separate result in full generality. The rather technical assumptions will become clear in the applications. As an immediate corollary we will obtain the aforementioned result of Sabiini, Theorem 1.3.

We will use the following convenient notation as in [Wol05]:

**Definition 2.1.** Let  $f_j: \mathbb{C}^n \rightarrow \mathbb{C}^n, j \in \mathbb{N}$ , be a sequence of holomorphic automorphisms. Then we set

$$f_{j,k} := \begin{cases} f_k \circ f_{k-1} \circ \cdots \circ f_{j+1} \circ f_j & \text{if } j \leq k, \\ \text{id} & \text{else} \end{cases}$$

and

$$f_{j,k}^{-1} := (f_{j,k})^{-1}.$$

**Proposition 2.2.** Let  $f_j: \mathbb{C}^n \rightarrow \mathbb{C}^n, j \in \mathbb{N}$ , be holomorphic automorphisms fixing 0. Assume there exist  $\delta > 0, 1 > r > 0$  and  $K \in \mathbb{N}$  such that for all  $z \in B_\delta$  and for all  $j, k \in \mathbb{N}$  with  $k \geq j + K - 1$  the following holds:

$$(3) \quad \|f_{j,k}(z)\| \leq r^{k-j+1} \|z\|$$

Moreover we assume that for each  $j \in \mathbb{N}$  there exist holomorphic automorphisms  $G_j: \mathbb{C}^n \rightarrow \mathbb{C}^n$  and holomorphic self-maps  $T_j: \mathbb{C}^n \rightarrow \mathbb{C}^n$  that satisfy the following:

$$(4) \quad T_j(0) = 0, \quad G_j(0) = 0$$

$$(5) \quad \forall j \in K\mathbb{N}_0 : \quad d_0 G_{j+1,j+K} = f_{j+1,j+K}, \quad d_0 T_{j+K} = \text{id},$$

$$(6) \quad G_{1,j} \rightrightarrows 0 \text{ on compacts for } j \rightarrow \infty$$

$$(7) \quad \exists b > 0 \forall z \in B_\delta \forall j \in K\mathbb{N} : \|T_j(z)\| \leq b \|z\|$$

We further assume that there exists an open neighborhood  $W$  of  $0 \in \mathbb{C}^n$  such that

$$(8) \quad \forall j \in K\mathbb{N} : T_j|_W \text{ is one-to-one}$$

and that there exist  $\rho > 0, \gamma > 0$  and  $a > 0$  such that

$$(9) \quad \forall z, z' \in B_\rho \forall j, k \in K\mathbb{N}, k > j : \\ \|G_{j+1,k}^{-1}(z) - G_{j+1,k}^{-1}(z')\| \leq a\gamma^{k-j} \|z - z'\|$$

and that there exist  $c > 0$  and  $q \in \mathbb{N}$  with  $r^q \gamma < 1$  and

$$(10) \quad \forall z \in B_\delta \forall j \in K\mathbb{N} : \\ \|G_{j+1,j+K}^{-1} \circ T_{j+K} \circ f_{j+1,j+K}(z) - T_j(z)\| \leq c \|z\|^q$$

Then the domain

$$\Omega := \left\{ z \in \mathbb{C}^n : \lim_{j \rightarrow \infty} f_{1,j}(z) = 0 \right\}$$

is biholomorphic to  $\mathbb{C}^n$ .

*Proof.* We may choose  $\delta > 0$  arbitrarily small. Hence w.l.o.g. let

$$(11) \quad b\delta < \rho.$$

By (3) we get

$$\forall j \geq K \forall z \in B_\delta : |f_{1,j}(z)| \leq r^j |z| < r^j \delta.$$

Hence we have uniform convergence

$$(12) \quad f_{1,j}|_{B_\delta} \rightrightarrows 0$$

and it follows that

$$\Omega \subset \bigcup_{j \in \mathbb{N}_0} f_{1,j}^{-1}(B_\delta).$$

Conversely, let  $j \in \mathbb{N}_0$  und  $z \in f_{1,j}^{-1}(B_\delta)$ . Then we have  $f_{1,j}(z) \in B_\delta$  and hence for  $k \geq j + K$

$$\|f_{1,k}(z)\| = \|f_{j+1,k}(f_{1,j}(z))\| \leq r^{k-j} \|f_{1,j}(z)\| \xrightarrow{k \rightarrow \infty} 0.$$

Altogether we obtain

$$(13) \quad \Omega = \bigcup_{j \in \mathbb{N}_0} f_{1,j}(B_\delta).$$

In particular we have that  $\Omega$  is an open and connected subset of  $\mathbb{C}^n$ . With (12) it follows in addition that

$$(14) \quad f_{1,j} \rightrightarrows 0 \text{ on compacts in } \Omega.$$

For  $l \in \mathbb{N}$  we define the sequence  $(\psi_j^l)_{j \geq l+K}$  of maps

$$\psi_j^l : B_\delta \rightarrow \mathbb{C}^n$$

by

$$\psi_j^l := G_{l+1,j}^{-1} \circ T_j \circ f_{l+1,j}.$$

Now let  $l, j \in K\mathbb{N}$  with  $j \geq l + K$ . From (3) and (7) we get for  $z \in B_\delta$  that

$$\begin{aligned} \|f_{l+1,j}(z)\| &\leq r^{j-l} \delta, \\ \|T_j \circ f_{l+1,j}(z)\| &\leq br^{j-l} \delta \end{aligned}$$

and

$$\|T_{j+K} \circ f_{j+1,j+K}(z)\| \leq br^K \|z\|.$$

And from (4) and (9) we get for  $z \in B_\rho$  that

$$\|G_{j+1,j+K}^{-1}(z)\| \leq a\gamma^K \|z\|.$$

Altogether we have for

$$\begin{aligned} \tilde{z} &:= G_{j+1,j+K}^{-1} \circ T_{j+K} \circ f_{j+1,j+K}(f_{l+1,j}(z)) \\ \hat{z} &:= T_j(f_{l+1,j}(z)), \end{aligned}$$

that

$$\exists J \in K\mathbb{N} \forall l, j \in K\mathbb{N}, j \geq l + J \forall z \in B_\delta : \|\tilde{z}\| \leq \rho, \|\hat{z}\| \leq \rho.$$

Hence together with (3), (9) and (10) we obtain

$$\begin{aligned}
 & \forall l, j \in K\mathbb{N}, j \geq l + J \forall z \in B_\delta : \\
 & \|\psi_{j+K}^l(x) - \psi_j^l(x)\| = \|G_{l+1,j}^{-1}(\tilde{z}) - G_{l+1,j}^{-1}(\hat{z})\| \\
 & \leq a\gamma^{j-l} \|\tilde{z} - \hat{z}\| \\
 (15) \quad & \leq a\gamma^{j-l} c \|f_{l+1,j}(z)\|^q \\
 & \leq a\gamma^{j-l} c (r^{j-l}\delta)^q \\
 & = ac\delta^q (r^q\gamma)^{j-l}.
 \end{aligned}$$

It follows that the subsequences  $(\psi_j^l)_{j \in K\mathbb{N}}$  converge uniformly (on  $B_\delta$ ) to maps  $\psi^l: B_\delta \rightarrow \mathbb{C}^n$ .

From (15) we also get (together with (11)) that

$$\begin{aligned}
 \|\psi^l\|_{B_\delta} &= \|\psi^l - \psi_{l+J}^l + \psi_{l+J}^l\|_{B_\delta} \\
 &\leq \sum_{\substack{j \in K\mathbb{N} \\ j \geq l+J}} \|\psi_{j+K}^l - \psi_j^l\|_{B_\delta} + \|\psi_{l+J}^l\|_{B_\delta} \\
 &< \sum_{\substack{j \in K\mathbb{N} \\ j \geq l+J}} ac\delta^q (r^q\gamma)^{j-l} + a\gamma^J br^J \delta \\
 &= \sum_{\substack{j \in K\mathbb{N} \\ j \geq J}} ac\delta^q (r^q\gamma)^j + a\gamma^J br^J \delta < \infty.
 \end{aligned}$$

This estimate does not depend on  $l$ . Hence we obtain

$$(16) \quad \exists R > 0 \forall l \in K\mathbb{N}: \quad \|\psi^l\|_{\varphi^{-1}(B_\delta)} < R.$$

Now we consider the sequence  $(\psi_j)_{j \in K\mathbb{N}}$  of holomorphic maps  $\psi_j: \Omega \rightarrow \mathbb{C}^n$  defined by

$$\psi_j := G_{1,j}^{-1} \circ T_j \circ f_{1,j}.$$

Let  $E \subset \Omega$  be a compact set. From (14) we get

$$\exists l \in K\mathbb{N}: f_{1,l}(E) \subset B_\delta.$$

Hence we have for  $j \in K\mathbb{N}$  with  $j \geq l + K$  that

$$(17) \quad \psi_j|_E = G_{1,l}^{-1} \circ \psi_j^l \circ f_{1,l}|_E$$

Therefore the uniform convergence of  $(\psi_j^l)_{j \in K\mathbb{N}}$  on  $B_\delta$  implies the uniform convergence of  $(\psi_j)_{j \in K\mathbb{N}}$  on  $E$ . Hence  $(\psi_j)_{j \in K\mathbb{N}}$  converges uniformly on compacts of  $\Omega$  to a holomorphic map  $\psi: \Omega \rightarrow \mathbb{C}^n$ .

From (5) we get

$$(18) \quad \det d_0 \psi = \det \text{id} = 1.$$

Let  $x, y \in \Omega$  with  $\psi(x) = \psi(y)$ . There exists a relatively compact, open and connected set  $E \subset \Omega$  with  $0, x, y \in E$ . By (14) it follows that

$$\exists L \in K\mathbb{N} \forall j \in K\mathbb{N}, j \geq L : f_{1,j}(E) \subset W.$$

By (8) we then know  $\psi_j$  to be one-to-one on  $E$  for such  $j$ . Then (18) implies that  $\psi$  is one-to-one on  $E$ . Hence  $x = y$  holds and we have shown that  $\psi$  is one-to-one.

For  $l \in K\mathbb{N}$  we have by (3) that

$$\exists J \in K\mathbb{N} \forall j \in K\mathbb{N}, j \geq J : f_{l+1,j}(B_\delta) \subset W.$$

Hence  $\psi_j^l$  is one-to-one for such  $j$ . Like (18) we also have

$$(19) \quad \det d_0 \psi^l \neq 0.$$

Hence  $\psi^l$  is one-to-one and open. By (16) and Montel's Theorem there exists a subsequence of  $(\psi^l)_{l \in K\mathbb{N}}$  converging to a holomorphic map  $\hat{\psi} : B_\delta \rightarrow \mathbb{C}^n$ . From (19) it follows  $\det d_0 \hat{\psi} \neq 0$ . Hence  $\hat{\psi}$  is also one-to-one. Altogether we have for  $0 < \tilde{\delta} < \hat{\delta} < \delta$  that

$$\exists L \in K\mathbb{N} \forall l \in K\mathbb{N}, l \geq L : \hat{\psi}(\overline{B_{\tilde{\delta}}}) \subset \psi^l(\overline{B_{\hat{\delta}}}).$$

Together with  $\hat{\psi}(0) = 0$  we obtain

$$(20) \quad \exists \varepsilon > 0 \forall l \in K\mathbb{N}, l \geq L : B_\varepsilon \subset \psi^l(\overline{B_{\hat{\delta}}}).$$

Let  $M > 0$  be arbitrarily large. By (6) there exists  $l \in K\mathbb{N}, l \geq L$ , s.t.  $G_{1,l}(B_M) \subset B_\varepsilon$ . Define a compact set  $E := f_{1,l}^{-1}(\overline{B_{\hat{\delta}}})$ . Then (13) implies

$$E \subset f_{1,l}^{-1}(B_\delta) \subset \Omega.$$

Because of  $f_{1,l}(E) \subset (B_\delta)$  we have by (17) that

$$\psi|_E = G_{1,l}^{-1} \circ \psi^l \circ f_{1,l}|_E.$$

Together with (20) we finally obtain

$$\begin{aligned} B_M \subset G_{1,l}^{-1}(B_\varepsilon) &\subset G_{1,l}^{-1}(\psi^l(\overline{B_{\hat{\delta}}})) = G_{1,l}^{-1}(\psi^l(f_{1,l}(E))) \\ &= \psi(E) \\ &\subset \psi(\Omega). \end{aligned}$$

Hence the image of  $\Omega$  under  $\psi$  is the whole of  $\mathbb{C}^n$ . □

*Proof of Theorem 1.3.* We use Proposition 2.2.

Let  $A_j := d_0 f_j$ . By (1) we then have

$$(21) \quad A_j^{-1} \circ f_j - \text{id} \in O(|\cdot|^p).$$

We define

$$\begin{aligned} T_j &:= \text{id} \\ G_j &:= A_j \end{aligned}$$

Clearly, the assumptions (4), (5), (7) and (8) of Proposition 2.2 are fulfilled.

We have  $\|G_j\| \leq r$  and it is easy to see that  $\|G_j^{-1}\| \leq \frac{1}{s}$ . Hence (6) holds and (9) holds for an arbitrary  $\rho > 0$ ,  $a = 1$  and  $\gamma = \frac{1}{s}$ . All  $G_j^{-1}$  and all  $f_j$  are uniformly bounded (on  $B_\delta$ ). Hence (21) implies (10).

Together with  $r^p \gamma = \frac{r^p}{s} < 1$  the theorem now follows from Proposition 2.2 using  $q = p$ .  $\square$

### 3. PROOFS

In order to prove Theorem 1.5 using Proposition 2.2 we will need the following lemmas with quantitative estimates. Therefore we will need some terminology introduced by Rosay–Rudin [RR88, Appendix].

Let  $c_1, \dots, c_n \in \mathbb{C}$  and for  $\nu \in \{2, \dots, n\}$  let  $h_\nu: \mathbb{C}^{\nu-1} \rightarrow \mathbb{C}$  be holomorphic maps with  $h_\nu(0) = 0$ . A holomorphic map  $G: \mathbb{C}^n \rightarrow \mathbb{C}^n$ ,  $G = (g_1, \dots, g_n)$ , is called *lower triangular*, if

$$\begin{aligned} g_1(z_1, \dots, z_n) &= c_1 z_1 \\ g_2(z_1, \dots, z_n) &= c_2 z_2 + h_2(z_1) \\ &\vdots \\ g_n(z_1, \dots, z_n) &= c_n z_n + h_n(z_1, \dots, z_{n-1}). \end{aligned}$$

It is called *polynomial lower triangular*, if all  $h_\nu$  are polynomial. In this case we define

$$\deg G := \max_{\nu \in \{1, \dots, n\}} \deg g_\nu.$$

Those elements  $c_\nu$  are called *diagonal elements*. It is easy to see that  $G$  is a holomorphic automorphism if and only if no  $c_\nu$  vanishes.

For  $m \in \mathbb{N}$ ,  $m \geq 2$ , let  $\mathcal{P}^m$  denote the vector space of all holomorphic maps  $h: \mathbb{C}^n \rightarrow \mathbb{C}^n$  whose components consist of homogeneous polynomials in  $n$  variables of order  $m$ . Let  $\lambda_1, \dots, \lambda_n$  denote the eigenvalues of a linear map  $A$  s.t.  $0 < |\lambda_n| \leq \dots \leq |\lambda_1| < 1$ . Clearly, the maps of the form  $h(z) = (0, \dots, 0, z^\alpha, 0, \dots, 0)$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $|\alpha| = m$ , provide a basis of  $\mathcal{P}^m$ . Such an  $h$  with all components but the  $j$ th vanishing is called *special (with respect to  $A$ )*, if  $\alpha_j = \dots = \alpha_n = 0$  and  $\lambda_j = \lambda_1^{\alpha_1} \dots \lambda_{j-1}^{\alpha_{j-1}}$ . We denote the vector subspace of those special elements by  $\mathcal{X}_A^m$ .

Rosay–Rudin [RR88, Appendix] observed the following: If we have  $|\lambda_1|^p < |\lambda_n|$  for some  $p \in \mathbb{N}$ ,  $p \geq 2$ , we get  $\mathcal{X}_A^m = 0$  for all  $m \geq p$ .

**Lemma 3.1.** *Let  $A \in \mathbb{C}^{n \times n}$  be lower triangular with eigenvalues  $1 > |\lambda_1| \geq \dots \geq |\lambda_n| > 0$  down its main diagonal and let  $m \in \mathbb{N}$ ,  $m \geq 2$ . Then for every  $R \in \mathcal{P}^m$  there exist  $X \in \mathcal{X}_A^m$  and  $H \in \mathcal{P}^m$  s.t.  $R = X + \Gamma_A H$  where  $\Gamma_A: \mathcal{P}^m \rightarrow \mathcal{P}^m$  is the commutator map  $\Gamma_A H := A \circ H - H \circ A$ . If  $A$  is diagonal, then  $X$  can be chosen to satisfy  $\|X\|_{\Delta_1} \leq nQ(m) \|R\|_{\Delta_1}$ .*

*Proof.* The proof of the general case (without estimate) is due to Rosay and Rudin [RR88, Appendix, Lemma 2]. They first consider our special



case  $A = \text{diag}(\lambda_1, \dots, \lambda_n)$ . In this case  $\Gamma_A$  is an invertible linear operator on the space of non-special elements of  $\mathcal{P}^m$ . Let  $P: \mathcal{P}^m \rightarrow \mathcal{X}_A^m$  be the projection onto  $\mathcal{X}_A^m$  and set  $X := PR$  and  $H := (\Gamma_A)^{-1}(R - PR)$ . Then the estimate follows from  $\|P\| \leq nQ(m)$ .  $\square$

*Remark 3.2.* In case of dimension  $n = 2$  the vector subspace  $\mathcal{X}_A^m$  is at most one-dimensional. In this case it is spanned by

$$(z_1, z_2) \mapsto (0, z_1^m).$$

For

$$R(z_1, z_2) = \left( \sum_{i=0}^m a_{1,i} z_1^i z_2^{m-i}, \sum_{i=0}^m a_{2,i} z_1^i z_2^{m-i} \right),$$

we then have:

$$\begin{aligned} \|X\|_{\Delta_1} &\leq \sup_{(z_1, z_2) \in \Delta_1} \left\| a_{2,m} \begin{pmatrix} 0 \\ z_1^m \end{pmatrix} \right\| \leq \sup_{\substack{(z_1, z_2) \in \Delta_1 \\ z_2=0}} \left\| \begin{pmatrix} \sum_{i=0}^m a_{1,i} z_1^i z_2^{m-i} \\ \sum_{i=0}^m a_{2,i} z_1^i z_2^{m-i} \end{pmatrix} \right\| \\ &\leq \|R\|_{\Delta_1} \end{aligned}$$

**Lemma 3.3.** *Let  $f: \mathbb{C}^n \rightarrow \mathbb{C}^n$  be a holomorphic map fixing 0 and  $A := d_0 f$  with eigenvalues  $0 < |\lambda_i| < 1$ .*

*Then there exist a unitary linear map  $S$ , a polynomial lower triangular automorphism  $\tilde{G}$  with  $\tilde{G}(0) = 0$  and  $d_0 \tilde{G} = S^{-1} \circ A \circ S$  and polynomials  $T^m: \mathbb{C}^n \rightarrow \mathbb{C}^n$ ,  $m \in \mathbb{N}$ ,  $m \geq 2$ , with  $T^m(0) = 0$  and  $d_0 T^m = \text{id}$  s.t.*

$$S \circ \tilde{G}^{-1} \circ S^{-1} \circ T^m \circ f - T^m \in O(|\cdot|^m).$$

*$S$  only depends on  $A$ . For  $m > 2$  all maps  $T^m$  only depend on the derivatives of  $f$  up to order  $m - 1$ . In addition we have  $T^2 = \text{id}$ . If all derivatives of  $f$  up to order  $m - 1$  are special respective  $A$ , we have  $T^m = \text{id}$ .*

*If for some  $p \in \mathbb{N}$ ,  $p \geq 2$ , we have  $\mathcal{X}_A^m = 0$  for all  $m \geq p$  then  $\tilde{G}$  only depends on the derivatives of  $f$  up to order  $p - 1$  and we have  $\deg \tilde{G} \leq p - 1$ . In particular we have  $\tilde{G} = S^{-1} \circ A \circ S$  for  $p = 2$ .*

*If  $A$  is normal and if we find  $0 < \delta \leq 1$  and  $0 < s < r$  s.t.*

$$(22) \quad \forall z \in B_\delta : s \|z\| \leq \|f(z)\| \leq r \|z\|,$$

*then we can write*

$$\tilde{G} = S^{-1} \circ A \circ S + H$$

*with*

$$\|S^{-1} \circ A^{-1} \circ S\| \leq \frac{1}{s}$$

*and*

$$\forall z \in B_{\frac{\delta}{\sqrt{n}}} : \|H(z)\| \leq r \|z\| \sum_{m=2}^{p-1} n^2 Q^2(m) \left( \frac{\sqrt{n}}{\delta} \right)^m.$$

*Proof.* We denote by  $\lambda_1, \dots, \lambda_n$  the eigenvalues of  $A$  s.t.  $0 \leq |\lambda_n| \leq \dots, \leq |\lambda_1| < 1$  and first consider the special case that  $A$  is lower triangular with  $\lambda_1, \dots, \lambda_n$  down its main diagonal. The proof (without estimates) for this case is due to Rosay and Rudin [RR88, Appendix, Lemma 3].

We recall from their proof the inductive construction of those polynomials  $T^m$  and polynomial lower triangular automorphisms  $\tilde{G}^m$  with  $\tilde{G}^m(0) = 0$  and  $d_0 \tilde{G}^m = A$  s.t.

$$(23) \quad T^m \circ f - \tilde{G}^m \circ T^m \in O(|\cdot|^m).$$

Let  $T^2 := \text{id}$  and  $\tilde{G}^2 := A$ . If for some  $m$  the maps  $T^m$  and  $\tilde{G}^m$  are constructed and (23) holds, then there exists  $R^m \in \mathcal{P}^m$  s.t.

$$T^m \circ f - \tilde{G}^m \circ T^m - R^m \in O(|\cdot|^{m+1}).$$

In the case that  $R^m$  is special, we set  $X^m := R^m$  and  $H^m := 0$ . Otherwise Lemma 3.1 gives us  $X^m \in \mathcal{X}_A^m$  and  $H^m \in \mathcal{P}^m$  with

$$R^m = X^m + A \circ H^m - H^m \circ A.$$

Note that if  $A$  is diagonal, we get (in both cases) the estimate

$$(24) \quad \|X^m\|_{\Delta_1} \leq nQ(m) \|R^m\|_{\Delta_1}.$$

Now let  $\tilde{G}^{m+1} := \tilde{G}^m + X^m$  and  $T^{m+1} := T^m + H^m \circ T^m$ . For  $m$  large enough we have  $\mathcal{X}_A^m = 0$  and hence  $\tilde{G}^m = \tilde{G}^{m+1} =: G$ . Those maps satisfy the desired properties with  $S = \text{id}$ .

To prove the general case we find a unitary  $S$  s.t.  $\tilde{A} := S^{-1} \circ A \circ S$  meets the requirements of the special case above. For  $\tilde{f} := S^{-1} \circ f \circ S$  we will then find maps  $\tilde{T}^m$  and  $\tilde{G}$  s.t.

$$\tilde{G}^{-1} \circ \tilde{T}^m \circ \tilde{f} - \tilde{T}^m \in O(|\cdot|^m).$$

With  $T^m := S \circ \tilde{T}^m \circ S^{-1}$  we can rewrite this to

$$S \circ \tilde{G}^{-1} \circ S^{-1} \circ T^m \circ f - T^m \in O(|\cdot|^m).$$

All formulated dependencies are obvious by construction.

If  $A$  is normal, we can choose  $S$  s.t.  $\tilde{A}$  is diagonal. The construction above yields

$$(25) \quad \tilde{G} = \tilde{A} + \sum_{m=2}^{p-1} \tilde{X}^m.$$

By (22) we have  $\|\tilde{A}^{-1}\| \leq \frac{1}{s}$ . All  $\tilde{X}^m$  and  $\tilde{R}^m$  are homogeneous polynomials. Hence from (24) follows  $\|\tilde{X}^m\|_{\Delta_{\frac{\delta}{\sqrt{n}}}} \leq nQ(m) \|\tilde{R}^m\|_{\Delta_{\frac{\delta}{\sqrt{n}}}}$ . By

the Cauchy integral formula we get

$$\left\| \tilde{R}^m \right\|_{\Delta_{\frac{\delta}{\sqrt{n}}}} \leq nQ(m) \frac{\|F\|_{\Delta_{\frac{\delta}{\sqrt{n}}}}}{\left(\frac{\delta}{\sqrt{n}}\right)^m}.$$

Together we have for all  $z \in B_{\frac{\delta}{\sqrt{n}}}$

$$\begin{aligned} \left\| \tilde{X}^m(z) \right\| &= \left\| \tilde{X}^m \left( \|z\| \frac{\sqrt{n}}{\delta} \frac{z}{\|z\|} \frac{\delta}{\sqrt{n}} \right) \right\| \\ &= \left( \|z\| \frac{\sqrt{n}}{\delta} \right)^m \left\| \tilde{X}^m \right\|_{\Delta_{\frac{\delta}{\sqrt{n}}}} \\ &\leq \|z\| \frac{\sqrt{n}}{\delta} n^2 Q^2(m) \left( \frac{\sqrt{n}}{\delta} \right)^m \|F\|_{\Delta_{\frac{\delta}{\sqrt{n}}}} \\ &\leq \|z\| n^2 Q^2(m) \left( \frac{\sqrt{n}}{\delta} \right)^m r. \end{aligned}$$

Hence for

$$H := \sum_{m=2}^{p-1} \tilde{X}^m,$$

we have

$$\forall z \in B_{\frac{\delta}{\sqrt{n}}} : \|H(z)\| \leq r \|z\| \sum_{m=2}^{p-1} n^2 Q^2(m) \left( \frac{\sqrt{n}}{\delta} \right)^m. \quad \square$$

*Remark 3.4.* In case of dimension  $n = 2$  there can be at most one  $m$  for which  $\mathcal{X}_A^m \neq 0$ . Hence at most one summand in (25) does not vanish. Together with Remark 3.2 we obtain a better estimate:

$$\forall z \in B_{\frac{\delta}{\sqrt{2}}} : \|H(z)\| \leq 2rQ(p-1) \left( \frac{\sqrt{2}}{\delta} \right)^{p-1} \|z\|.$$

**Lemma 3.5.** *Let  $G$  be a lower triangular holomorphic automorphism with  $\deg G \leq d$  for some  $d \in \mathbb{N}$  and*

$$\forall z \in B_\rho : \|G(z)\| \leq C \|z\|$$

*for some  $0 < \rho < 1$  and  $C > 0$ . Then there exists  $\gamma > 0$  s.t.*

$$G_{1,k} \left( \Delta_{\frac{\rho}{\sqrt{n}}} \right) \subset \Delta_{\gamma^k}$$

*where  $\Delta_\delta$  is the polydisc of radius  $\delta$  about 0. We may choose*

$$(26) \quad \gamma := \sum_{i=1}^{d^{n-1}} Q(i) (\sqrt{n})^i C^i.$$

*Proof.* The proof without estimate is due to Rosay and Rudin [RR88, Appendix, Lemma 1]. They first observed

$$(27) \quad \forall k \in \mathbb{N} : \deg G_{1,k} \leq d^{n-1}.$$

We denote by  $(G)_\nu$  the  $\nu$ -th component of  $G$ . Then we have

$$(28) \quad \forall z \in \Delta_{\frac{\rho}{\sqrt{n}}} : |(G(z))_\nu| \leq C\rho \leq \gamma.$$

Now suppose that

$$\forall z \in \Delta_{\frac{\rho}{\sqrt{n}}} : |(G_{1,k})_\nu| \leq \gamma^k$$

for some  $k \in \mathbb{N}$ . By (27) we have that

$$(G_{1,k}(z))_\nu = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ 1 \leq |\alpha| \leq d^{n-1}}} a_{k,\nu,\alpha} z^\alpha$$

and by the Cauchy integral formula we get

$$|a_{k,\nu,\alpha}| \leq \frac{\gamma^k}{\rho^{|\alpha|}}$$

Together with  $G_{1,k+1} = G_{1,k} \circ G$  and (28) we obtain for  $z \in \Delta_{\frac{\rho}{\sqrt{n}}}$

$$\begin{aligned} |(G_{1,k+1}(z))_\nu| &= \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ 1 \leq |\alpha| \leq d^{n-1}}} a_{k,\nu,\alpha} (G(z))_1^{\alpha_1} \cdots (G(z))_n^{\alpha_n} \\ &\leq \sum_{i=1}^{d^{n-1}} \frac{\gamma^k}{\rho^i} (\sqrt{n})^i Q(i) C^i \rho^i \\ &\leq \gamma^{k+1}. \end{aligned} \quad \square$$

**Lemma 3.6.** [RR88, Appendix, Lemma 1 (b)] *Let  $G$  be a lower triangular holomorphic automorphism with diagonal elements  $|c_\nu| < 1$ . Then we have*

$$G_{1,j} \rightrightarrows 0 \text{ on compacts for } j \rightarrow \infty.$$

**Lemma 3.7.** *Let  $G: \mathbb{C}^n \rightarrow \mathbb{C}^n$  be a lower triangular holomorphic automorphism with  $G = A + H$  where  $A := d_0 G$  and  $H: \mathbb{C}^n \rightarrow \mathbb{C}^n$  is a holomorphic self-map. We assume that there exist  $0 < s < 1$ ,  $\delta > 0$  and  $C \geq 0$  with*

$$\|A^{-1}\| \leq \frac{1}{s}$$

and

$$\forall z \in B_\delta : \|H(z)\| \leq C \|z\|.$$

Then

$$\forall z \in B_{\frac{\delta s^n}{\sqrt{n}(n-1)!(1+C)^{n-1}}} : \|G^{-1}(z)\| \leq \sqrt{n} (n-1)! \frac{(1+C)^{n-1}}{s^n} \|z\|.$$

*Proof.* Let  $A^{-1} = (A_1^{-1}, \dots, A_n^{-1})$  and  $H = (H_1, \dots, H_n)$ . With  $G$  also  $G^{-1}$  is lower triangular. Therefore  $A^{-1}$  is lower triangular. Hence for  $\nu \in \{1, \dots, n\}$  we have

$$\begin{aligned}
 (29) \quad \forall z \in \mathbb{C}^n : \quad & |A_\nu^{-1}(z)| = |A_\nu^{-1}(z_1, \dots, z_\nu, 0, \dots, 0)| \\
 & \leq \|A^{-1}(z_1, \dots, z_\nu, 0, \dots, 0)\| \\
 & \leq \frac{1}{s} \|(z_1, \dots, z_\nu, 0, \dots, 0)\|.
 \end{aligned}$$

$G$  is lower triangular and therefore  $H = G - d_0 G$  is a lower triangular map with vanishing diagonal elements. Hence for  $\nu \in \{1, \dots, n\}$  we have

$$\begin{aligned}
 (30) \quad \forall z \in B_\delta : \quad & |H_\nu(z)| = |H_\nu(z_1, \dots, z_{\nu-1}, 0, \dots, 0)| \\
 & \leq \|H(z_1, \dots, z_{\nu-1}, 0, \dots, 0)\| \\
 & \leq C \|(z_1, \dots, z_{\nu-1}, 0, \dots, 0)\|.
 \end{aligned}$$

Let  $z \in \mathbb{C}^n$  with  $\|z\| \leq \frac{\delta s^n}{\sqrt{n(n-1)!(1+C)^{n-1}}}$ . For  $z = G(w) = A(w) + H(w)$  we have  $G^{-1}(z) = w = A^{-1}(z - H(w))$ . By (29) and (30) it follows that

$$|w_1| \leq \frac{1}{s} |z_1| + 0 = (1-1)! \frac{(1+C)^{1-1}}{s^1} \sqrt{\sum_{\eta=1}^1 |z_\eta|^2}.$$

For  $\nu \in \{2, \dots, n\}$  assume that

$$|w_\mu| \leq (\mu-1)! \frac{(1+C)^{\mu-1}}{s^\mu} \sqrt{\sum_{\eta=1}^\mu |z_\eta|^2}$$

for all  $\mu \in \{1, \dots, \nu-1\}$ .

Together with (29) and (30) and noting that  $s < 1$  and  $1+C \geq 1$  we finally obtain the following estimates:

$$\begin{aligned}
|w_\nu| &\leq \frac{1}{s} \left( \sqrt{\sum_{\eta=1}^{\nu} |z_\eta|^2} + \sqrt{\sum_{\eta=1}^{\nu} |H_\eta(w)|^2} \right) \\
&\leq \frac{1}{s} \left( \sqrt{\sum_{\eta=1}^{\nu} |z_\eta|^2} + C\sqrt{\nu-1} \sqrt{\sum_{\eta=1}^{\nu-1} |w_\eta|^2} \right) \\
&\leq \frac{1}{s} \left( \sqrt{\sum_{\eta=1}^{\nu} |z_\eta|^2} + C(\nu-1) \sqrt{\left( (\nu-2)! \frac{(1+C)^{\nu-2}}{s^{\nu-1}} \right)^2 \sum_{\eta=1}^{\nu-1} |z_\eta|^2} \right) \\
&\leq \frac{1}{s} \left( \sqrt{\sum_{\eta=1}^{\nu} |z_\eta|^2} + C(\nu-1)! \frac{(1+C)^{\nu-2}}{s^{\nu-1}} \sqrt{\sum_{\eta=1}^{\nu} |z_\eta|^2} \right) \\
&\leq (\nu-1)! \frac{(1+C)^{\nu-2}}{s^\nu} \left( (1+C) \sqrt{\sum_{\eta=1}^{\nu} |z_\eta|^2} \right) \\
&= (\nu-1)! \frac{(1+C)^{\nu-1}}{s^\nu} \sqrt{\sum_{\eta=1}^{\nu} |z_\eta|^2}
\end{aligned}$$

In particular we have shown the desired estimate by induction.  $\square$

*Proof of Theorem 1.5.* We use Proposition 2.2. W.l.o.g. let  $z_0 = 0$ .

It is easy to see that every eigenvalue  $\lambda$  of  $d_0 f_j$  satisfies

$$(31) \quad 0 < s \leq |\lambda| \leq r < 1.$$

For  $j \in \mathbb{N}$  Lemma 3.3 gives us a unitary linear map  $S_j$ , a lower triangular automorphism  $\tilde{G}_j$  and holomorphic maps  $T_j^m$  with

$$(32) \quad S_j \circ \tilde{G}_j^{-1} \circ S_j^{-1} \circ T_j^m \circ f_j - T_j^m \in O(|\cdot|^m).$$

If  $q \geq 2$ , we have (according to Lemma 3.3) by (2) that all  $S_j =: S$  are identical. For  $j \in \mathbb{N}$  we define

$$(33) \quad G_j := S \circ \tilde{G}_j \circ S^{-1}$$

which fulfills (4) and (5) of Proposition 2.2. If  $q \geq p$ , we have (again, according to Lemma 3.3) by (31) that all  $\tilde{G}_j$  are identical. With  $\tilde{G}_j$  also  $\tilde{G}_j^{-1}$  is lower triangular. Hence (noting that  $S$  is unitary) Lemma 3.5 gives us  $0 < \tilde{\rho} < 1$  and  $\gamma > 0$  s.t.

$$G_{j+1,k}^{-1} \left( \Delta_{\frac{\rho}{\sqrt{n}}} \right) \subset \Delta_{\gamma^{k-j}}$$

for  $j > k$ . Application of the Schwarz–Pick Lemma then gives us some  $\rho > 0$  and  $a > 0$  s.t. (9) holds (with  $\gamma > 0$  from above). By Lemma 3.6 and (31) we get that (6) is fulfilled. If  $d_0 f_j$  is normal, then the application of Lemma 3.3 gives a linear estimate for  $\tilde{G}_j$  and

$\deg \tilde{G}_j \leq p - 1$ . Lemma 3.7 then gives a linear estimate for  $\tilde{G}_j^{-1}$  and it is easy to see that we have  $\deg \tilde{G}_j^{-1} \leq (p - 1)^{n-1}$ . Hence by the application of Lemma 3.5 we see that we may choose

$$(34) \quad \gamma = \sum_{i=1}^{(p-1)^{(n-1)^2}} (Q(i) \sqrt{n})^i \left( \sqrt{n} (n-1)! \frac{(1+C)^{n-1}}{s^n} \right)^i$$

where

$$(35) \quad C = r \sum_{m=2}^{p-1} n^2 Q^2(m) \left( \frac{\sqrt{n}}{\min\{1, \delta\}} \right)^m.$$

We choose  $q$  ( $\geq p$ ) large enough to satisfy  $r^q \gamma < 1$ . Then we define

$$T_j := T_j^q.$$

This fulfills (4) and (5) of Proposition 2.2. By (2) we have (according to Lemma 3.3) that all  $T_j$  are identical. Hence (7) and (8) hold. All maps in (32) are uniformly bounded (on  $B_\delta$ ) and hence (10) is fulfilled.

The theorem now follows from Proposition 2.2. If  $d_0 F_j$  is normal, the desired estimate follows from  $r^q \gamma < 1$  by (34). If in addition  $n = 2$ , the desired estimate follows from Remark 3.4. In both cases the estimates satisfy  $q \geq p \geq 2$  which is needed above.  $\square$

*Remark 3.8.* The goal in the proof above is to make sure all  $G_j$  and all  $T_j$  are identical. The assumptions in Theorem 1.5 are one way to achieve this. There are other possibilities:

- (1) If all derivatives of  $f_j$  up to order  $q - 1$  (like in the proof above) are special elements with respect to  $d_{z_0} f_j$ , Lemma 3.3 assures  $T_j^m = \text{id}$ . Then we need the assumption (2) of Theorem 1.5 just for multi-indices up to order  $p - 1$  in order to get  $G_1 = G_2 = \dots$  (according to Lemma 3.3).
- (2) If we have  $\mathcal{X}_{A_j}^m = 0$  for all  $m \in \{2, \dots, p - 1\}$  and  $A_j := d_{z_0} f_j$ , Lemma 3.3 gives us  $G_j = A_j$ . Then we may choose (in the proof above)  $\gamma = \frac{1}{s}$  and therefore  $q = p$ .

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